Is Commodity Taxation Unfair?

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Abstract

In a model where agents have unequal skills and heterogeneous preferences about consumption goods and leisure, this paper studies how to combine linear commodity taxes and non-linear income tax. It proposes a particular social welfare function on the basis of fairness principles. It then derives a simple criterion for evaluating the social welfare consequences of various tax schedules. Under the proposed approach, the optimal tax should have no commodity tax for some range of consumptions, and income redistribution would feature high subsidies to the working poor. It is also shown that, even when the income tax fails to be optimal, commodity taxes may not improve social welfare.

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1 Introduction

This paper studies how the combination of an income tax and commodity taxes has welfare and fairness consequences over a population which is heterogenous in preferences and earning abilities. It is well known that heterogeneity in individual preferences makes it difficult to define a social welfare function in a precise way, and that a double heterogeneity makes it hard to study the incentive-compatible allocation of resources. The first contribution of the paper is to propose a definition of social welfare which incorporates fairness criteria and which relies on such criteria in order to compare the situation of individuals with different preferences. The sole information about individuals that is considered by such social preferences is their ordinal non-comparable preferences. The interpersonal comparisons performed by these social preferences bear on individual indifference surfaces, and there is no need for additional information, of a comparable kind, about subjective utility.

The second contribution of the paper is to propose simple criteria for the normative comparison of tax schedules in view of their social consequences. Such simple decision

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criteria need very little information (sometimes none at all) about the population characteristics and can therefore be easily used by policy-makers. They can be obtained in spite of the double heterogeneity of agents, because the difficulty of multi-dimensional screening has to do with checking the feasibility of incentive-compatible allocations, not with the normative comparison of given (feasible or infeasible) allocations. One may argue that policy-makers need such simple criteria more than a theoretical description of the optimal tax scheme.

Nonetheless, the paper also explores the issue of the optimal tax. In a landmark contribution to this topic, Atkinson and Stiglitz (1976) showed that when non-linear income taxation is possible, a non-uniform commodity tax may be useless if individual preferences over consumption goods are separable from labor. This result is striking, although it is sometimes considered to have limited practical relevance, for several reasons. In particular, it holds only when the income tax is optimal. If the income tax is suboptimal, commodity taxation retains a useful role, as exemplified by the Diamond-Mirrlees-Samuelson formula (Samuelson 1951\(^1\), Diamond and Mirrlees 1971) obtained for the case of linear income taxation. Since actual income tax schedules, although non-linear, are unlikely to approximate the optimal schedules, differentiated commodity taxes may be useful in order to improve social welfare. Besides, the Atkinson-Stiglitz result holds only when individuals have the same preferences and differ only in the dimension of productivity. Additional sources of heterogeneity may destroy the result, as shown for instance in Cremer, Pestieau and Rochet (2001) for the case of unequal inherited wealth. Heterogenous consumer preferences are also likely to complicate the matter considerably, although this is seldom studied in the optimal tax literature because of the difficulty to define a sensible social welfare function over different utility functions.\(^2\) A third limitation is that the Atkinson-Stiglitz result crucially depends on separability of consumption and labor in individual preferences. In absence of such separability, it is better to put a greater tax on goods which are complementary to leisure (Christiansen 1984).

With the particular social preferences constructed here, this paper obtains results which have some similarity with the Atkinson-Stiglitz result. The main result along this vein is that, if non-linear commodity taxes are considered, then the working poor should be submitted to a uniform or null commodity tax. It is even shown that, among linear

\(^{1}\)This source is quoted in Atkinson and Stiglitz (1980), p. 373.

\(^{2}\)With respect to income taxation, see e.g. Boadway et al. (2002), in which various individual weights are considered, and Choné and Laroque (2001), in which an ethical assumption is made in order to normalize utility functions.
commodity cases, a non-uniform commodity tax may in some cases be unable to improve social welfare substantially, even when income taxation is not optimal and when individuals have heterogenous preferences in addition to unequal skills. These results do not involve any separability assumption about preferences, and the first one in particular holds for a wide class of population profiles.

This paper is a direct follow-up to Fleurbaey and Maniquet (2002a), where the focus was on income taxation, in a simple model with only one consumption good. Here several consumption goods are introduced, and the fairness criteria are extended to the multiple commodity case. In the sequel, after a presentation of the model and the fairness requirements (section 2), an axiomatic derivation of social preferences is made (section 3), which are then used as a decision criterion for the analysis of the combination of direct and indirect taxes (section 4). The last section concludes. The proofs are gathered in the appendix.

2 Model and definitions

2.1 The model

The model contains \( G + 1 \) goods, namely labor and \( G \) consumption goods. A bundle for agent \( i \) is a pair \( z_i = (\ell_i, c_i) \), where \( \ell_i \) is labor and \( c_i \) a \( G \)-dimensional consumption vector. The agents’ consumption set \( X \) is defined by the conditions \( 0 \leq \ell_i \leq 1 \) and \( c_i \geq 0 \).\(^3\) The zero vector of \( \mathbb{R}^{G+1} \) is denoted \( 0 \).

The population contains \( n \geq 2 \) agents. Agents have two characteristics, their personal preferences over the consumption set and their personal skill. For any agent \( i = 1, \ldots, n \), personal preferences are denoted \( R_i \), and \( z_i R_i z'_i \) (resp. \( z_i P_i z'_i \), \( z_i I_i z'_i \)) means that bundle \( z_i \) is weakly preferred (resp. strictly preferred, indifferent) to bundle \( z'_i \). Individual preferences are assumed to be continuous, convex and monotonic.\(^4\)

The technology is assumed to exhibit constant returns to scale, so that producer prices for consumption goods, expressed in the numeraire, are a fixed vector\(^5\) \( p \in \mathbb{R}^G_+ \). The numeraire is “efficient” labor. Because of their different skills, individual agents are

\(^3\)Vector inequalities are denoted \( \geq, >, \gg \).

\(^4\)Preferences are monotonic if \( \ell_i \leq \ell'_i \) and \( c_i > c'_i \) implies that \( (\ell_i, c_i) P_i (\ell'_i, c'_i) \).

\(^5\)By an appropriate choice of units of measurement for quantities of goods, prices could be normalized to 1 without loss of generality. A vector \( p \) is retained here for a more intuitive and explicit presentation of the role of prices in the definition of social preferences.
unequally able to provide efficient labor. Agent $i$’s productivity or wage rate, denoted $w_i$, is measured in units of the numeraire, so that $w_i \geq 0$ is the amount of efficient labor that agent $i$ provides when working $\ell_i = 1$, and, for any $\ell_i$, $w_i\ell_i$ is the agent’s pre-tax income (earnings).

An allocation is a collection $z = (z_1, \ldots, z_n)$. Social preferences will allow us to compare allocations in terms of their goodness. Social preferences will be formalized as a complete ordering over all allocations in $X^n$, and will be denoted $R$, with asymmetric and symmetric components $P$ and $I$, respectively. In other words, $z R z'$ means that $z$ is at least as good as $z'$, $z P z'$ means that it is strictly better, and $z I z'$ that they are equivalent.

Social preferences will typically depend on the population profile of characteristics $(R_1, \ldots, R_n)$ and $(w_1, \ldots, w_n)$, as in the theory of social choice. Formally, they are a mapping from the set of population profiles to the set of complete orderings over allocations. Special notations for these notions are not introduced, in order to minimize the quantity of symbols in this paper. The domain of economies for which social preferences are to be defined contains all economies obeying the above conditions.

### 2.2 Fairness requirements

The fairness requirements imposed on social preferences are essentially the same as in Fleurbaey and Maniquet (2002a), extended to the multi-dimensional case for consumption goods.

The first requirement is inspired by the Pigou-Dalton transfer principle, which plays a prominent role in the literature on inequality measurement.\(^6\) In contrast to this literature where transfers are deemed legitimate for all income inequalities, here the transfer principle is applied in a very cautious way. Indeed, in an axiomatic analysis one wants to impose axioms which are as little controversial as possible. For a transfer principle, this means that we want to declare a transfer desirable only in a case when there is absolute consensus that the donor has a better situation than the receiver. The following axiom says that a transfer of consumption is socially acceptable (not necessarily a strict social improvement) when the donor has the same quantity of labor and the same preferences as the receiver, and consumes more of each good.\(^7\)

**Transfer Principle:** If $z$ and $z'$ are two allocations, and $i$ and $j$ are two agents with

\(^6\)See e.g. Lambert (1989).

\(^7\)This does not mean that a transfer is undesirable in other contexts. The axiom only says that at least in this transparent case, social preferences should be correct.
identical preferences, such that \( \ell_i = \ell_j = \ell'_i = \ell'_j \), and for some \( \delta > 0 \),

\[
c_i' - \delta = c_i \gg c_j = c_j' + \delta,
\]

whereas for all other agents \( k, z_k = z'_k \), then \( z R z' \).

The second requirement goes in the direction of limiting redistribution. It relies on the idea that when all agents have the same wage rate, there is no need for redistribution, since they all have access to the same labor-consumption bundles. Any income and consumption difference is then a matter of personal preferences. A laisser-faire allocation \( z^* \) is such that for every agent \( i \), \( z^*_i \) is the best for \( R_i \) over the budget set defined by \( pc_i \leq w_i \ell_i \). The following requirement says that a laisser-faire allocation,\(^8\) in this particular case of uniform earning ability, is (one of) the best among all feasible allocations.

**Laisser-Faire:** If all agents have the same wage rate \( w \), then for any allocation \( z' \) such that \( p \sum_i c'_i \leq w \sum_i \ell'_i \), one has \( z^* R z' \), where \( z^* \) denotes a laisser-faire allocation.

The other requirements are basic conditions derived from the theory of social choice. First, the standard Pareto condition is essential in order to take account of efficiency considerations. Social preferences satisfying the Pareto condition will never lead to the selection of inefficient allocations.

**Weak Pareto:** If \( z \) and \( z' \) are such that for all \( i, z_i P z'_i \), then \( z P z' \).

Another basic condition, in the spirit of Arrow’s condition of independence of irrelevant alternatives,\(^9\) limits the amount of information about individual preferences that may be used in the comparison of two allocations. Introduced by Hansson (1973) and Pazner (1979) it requires social preferences over two allocations to depend only on individual indifference curves at these two allocations. This condition is satisfied by all criteria of fairness and cost-benefit analysis.

**Hansson Independence:** Let \( z \) and \( z' \) be two allocations, and \( R, R' \) be the social orderings for two profiles \((R_1, ..., R_n)\) and \((R'_1, ..., R'_n)\) respectively. If for all \( i \), and all \( q \in X \),

\[
\begin{align*}
z_i I_i q & \iff z_i I'_i q \\
z'_i I_i q & \iff z'_i I'_i q,
\end{align*}
\]

\(^8\)There may be several laisser-faire allocations if preferences are not strictly convex. But all laisser-faire allocations, in a given economy, give agents the same satisfaction.

\(^9\)Arrow’s condition is much too restrictive and leads to his impossibility theorem. For explanations of how excessive Arrow’s independence is, see Fleurbaey and Maniquet (1996, 2001) and Fleurbaey, Suzumura and Tadenuma (2003).
then

\[ z \ R \ z' \iff z \ R' \ z'. \]

A last requirement is that social preferences should have a separable structure. The intuition is that agents who are not concerned by a social decision need not be given any say in it. Hence the following condition, which says that an individual who has the same bundle in two allocations could be removed from the population without affecting the social judgment over these two allocations.

**Separability:** Let \( z \) and \( z' \) be two allocations, and \( i \) an agent such that \( z_i = z'_i \). Then

\[ z \ R \ z' \Rightarrow z_{-i} \ R_{-i} \ z'_{-i}, \]

where \( z_{-i} = (z_1, \ldots, z_{i-1}, z_{i+1}, \ldots, z_n) \), and \( R_{-i} \) is the social preference ordering for the economy with reduced population \( \{1, \ldots, i-1, i+1, \ldots, n\} \).

### 3 Social preferences

The combination of the above requirements for social preferences yields the following theorem, which gives a partial but useful description of social preferences.

**Theorem 1** If social preferences satisfy Transfer Principle, Laisser-Faire, Weak Pareto, Hansson Independence and Separability, then for any allocations \( z, z' \) such that \( z_i \ P_i 0 \) and \( z'_i \ R_i 0 \) for all \( i \), one has

\[ \min_i W_i(z_i) > \min_i W_i(z'_i) \Rightarrow z \ P \ z', \]

where \( W_i(z_i) = \max \{ w \in \mathbb{R}_+ \mid \forall (\ell, c) \in X \text{ s.t. } pc \leq w\ell, \ z_i R_i (\ell, c) \} \).

Concretely, \( W_i(z_i) \) is the wage rate which would enable agent \( i \) to reach the same satisfaction as in \( z_i \), if she were allowed to choose her labor time freely, at this wage rate, and to buy consumption goods at producer prices: “What tax-free wage rate would give you the same satisfaction as your current situation, in absence of commodity taxation?”  This question cannot be used as a practical device for assessing individuals’ situations, since they may have a hard time figuring out what the true answer is and, above all, would be tempted to misrepresent their situation. The next section will examine how the measurement of \( \min_i W_i(z_i) \) can be practically made by a government knowing only the statistical distribution of characteristics in the population.
A slightly different interpretation of $W_i(z_i)$ is possible. Consider an agent $i$ who is indifferent between $z_i$ and the bundle $z_i^*$ she would choose in a laisser-faire allocation that would be, according to Laisser-Faire, socially optimal if all agents had an equal wage rate $w^*$. Then $W_i(z_i) = w^*$. In other words, $W_i(z_i)$ is the hypothetical common wage rate which would render this agent indifferent between $z_i$ and an optimal allocation.

The function $W_i(z_i)$ is a particular money-metric utility representation of agent $i$’s preferences (for a part of the consumption set — see below). It makes it possible to compare the situations of individuals who have identical or different preferences, on the basis of their current indifference curves. In addition, the social preferences described in Theorem 1 give absolute priority to agents with the lowest $W_i(z_i)$. In this way, this result suggests a solution to the problem of weighting different utility functions, which was mentioned in the introduction.

The above theorem does not give a full characterization of social preferences, but this is not problematic for the purpose of tax analysis. The fact that it does not explain how to compare allocations for which $\min_i W_i(z_i) = \min_i W_i(z_j^*)$ is not important, since the most relevant insight is that a greater $\min_i W_i(z_i)$ means a greater social welfare. And the fact that the theorem is silent about allocations such that $0 P_i z_i$ for some agent $i$ is not a serious limitations since such allocations can be proved to be bad anyway (see Lemma 8 in the appendix, for details). As a consequence the theorem gives us essentially all one needs to study tax redistribution.

The intuitive proof of Th. 1 in Fleurbaey and Maniquet (2002a) can provide most of what is needed to understand the gist of the above result. The additional complication introduced by the presence of several consumption goods does not entail any important change in the structure of the main argument. When two agents $i$ and $j$, in two allocations $z$ and $z'$, are such that

$$W_i(z'_i) > W_i(z_i) > W_j(z_j) > W_j(z'_j),$$

and $z_k = z_k'$ for $k \neq i, j$, one can still show that one must have $z \succ z'$, by relying on hypothetical agents $a$ and $b$, such that $R_a = R_i$ and $R_b = R_j$, and $w_a = w_b = w$ with

$$W_i(z_i) > w > W_j(z_j).$$

Indeed, in a laisser-faire allocation $z^*$ for the $\{a, b\}$ population, one has $W_a(z_a^*) = W_b(z_b^*) = w$, so that $z_i P_i z_a^*$ and $z_b^* P_j z_j$. For $\varepsilon \in \{0\} \times \mathbb{R}_{++}^+$ small enough, one has $z'_i P_i z_i P_i z_a^* + \varepsilon P_i z_a^*$ and $z_b^* P_j z_j P_j z'_j$. In the $\{i, j, a, b\}$ population, one can then show that,

$$(z_i, z_j, z_a^* + \varepsilon, z_b^* - 2\varepsilon) \succ (z'_i, z'_j, z_a^*, z_b^*)$$
(by Weak Pareto, Transfer Principle and Hansson Independence —see Lemma 3 in the appendix for details). Notice that \((z^*_a + \varepsilon, z^*_b - 2\varepsilon)\) is a feasible and inefficient allocation in the \(\{a, b\}\) population, since 
\[z^*_a + \varepsilon + z^*_b - 2\varepsilon = z^*_a + z^*_b - \varepsilon < z^*_a + z^*_b.\]
By Laisser-Faire and Weak Pareto, therefore, \((z^*_a, z^*_b) \not\sim (z^*_a + \varepsilon, z^*_b - 2\varepsilon)\), and by Separability,
\[(z_i, z_j, z^*_a, z^*_b) \not\sim (z_i, z_j, z^*_a + \varepsilon, z^*_b - 2\varepsilon),\]
so that, by transitivity,
\[(z_i, z_j, z^*_a, z^*_b) \not\sim (z'_i, z'_j, z^*_a, z^*_b)\]
implying, by Separability, \((z_i, z_j) \not\sim (z'_i, z'_j)\) and therefore \(z \not\sim z'\).

4 Taxation

4.1 Setting

It is assumed here that taxation operates via an income tax function \(\tau : \mathbb{R}_+ \rightarrow \mathbb{R}\) and a commodity tax function \(\beta : \mathbb{R}_+^G \rightarrow \mathbb{R}\). Individual \(i\) then pays an income tax equal to \(\tau(y_i)\), where \(y_i = w_i\ell_i\), and a commodity tax \(\beta(c_i)\). Such taxes are subsidies when they are negative. It is not assumed a priori that commodity taxation is linear. A linear commodity tax is such that 
\[\beta(c_i) = \sum_{k=1}^G \varsigma_k p_k c_{ik},\]
where \(\varsigma_k\) denotes the tax rate for good \(k\). Even though general commodity taxes are considered here, we restrict attention, for the sake of realism, to taxes such that \(\beta(0) = 0\) and \(\beta(c)\) is non-decreasing in \(c\).

As is standard in the tax literature since Mirrlees (1971), individuals are free to choose their labor time and consumption in the budget set modified by the tax schedule. The government is assumed to know the distribution of types (preferences, earning abilities) in the population but ignores the characteristics of any particular agent. Since it is easy to forecast the behavior of any given type of agent under a tax schedule, knowing the distribution of types enables the government to forecast the social consequences of any tax schedule. It may then evaluate or choose a tax function in view of the foreseen social consequences.

Under the tax schedule \((\tau, \beta)\), agent \(i\)’s budget constraint is
\[pc_i + \beta(c_i) \leq w_i\ell_i - \tau(w_i\ell_i).\]
It is convenient to focus on the earnings-consumption space, in which the budget constraint is defined by
\[pc_i + \beta(c_i) \leq y_i - \tau(y_i).\]
The latter inequality is the same for all agents, except that for an agent with wage rate \( w_i \), necessarily \( y_i \leq w_i \). As a consequence, the budget set of an agent with greater skill is always bigger for inclusion.

In the earnings-consumption space, one can define individual preferences \( R_i^* \) over earnings-consumption bundles, and they are derived from ordinary preferences over labor-consumption bundles by:

\[
(y, c) R_i^*(c', y') \iff \left( \frac{y}{w_i}, c \right) R_i \left( \frac{y'}{w_i}, c' \right).
\]

The fact that all agents are submitted to the same constraint \( pc + \beta(c) \leq y - \tau(y) \) implies that for any pair of agents \( i, j \), when \( i \) chooses \( (y_i, c_i) \) and \( j \) chooses \( (y_j, c_j) \), one must have \( (y_i, c_i) R_i^*(y_j, c_j) \) or \( y_j > w_i \). Conversely, any allocation \( z \) satisfying the self-selection constraints:

\[
\text{for all } i, j, \ (y_i, c_i) R_i^*(y_j, c_j) \text{ or } y_j > w_i
\]

is incentive-compatible and can be obtained by letting every agent \( i \) choose her best bundle under a budget constraint \( pc + \beta(c) \leq y - \tau(y) \) for some well-chosen tax schedule \((\tau, \beta)\). This tax schedule must be such that the surface

\[
S(\tau, \beta) = \{(y, c) \in \mathbb{R}_+^{G+1} \mid pc + \beta(c) = y - \tau(y)\}
\]

lies nowhere above the envelope curve of the indifference surfaces of the population in the \((y, c)\)-space, and intersects this envelope curve at all points \((y_i, c_i)\) for \( i = 1, ..., n \).

By monotonicity of individual preferences, we may restrict our attention to tax schedules \((\tau, \beta)\) such that for any \( y < y' \),

\[
\{ c \in \mathbb{R}_+^G \mid pc + \beta(c) \leq y - \tau(y) \} \subseteq \{ c \in \mathbb{R}_+^G \mid pc + \beta(c) \leq y' - \tau(y') \}.
\]

An allocation is \textit{feasible} if it satisfies

\[
p \sum_{i=1}^n c_i \leq \sum_{i=1}^n y_i.
\]

A tax function \( \tau \) is \textit{feasible} if it satisfies

\[
\sum_{i=1}^n \left[ \tau(w_i \ell_i + \beta(c_i)) \right] \geq 0
\]

when all agents choose their labor time and consumption by maximizing their satisfaction over their budget set.
A tax schedule \((\tau, \beta)\) will be called minimal when any tax schedule \((\tau', \beta')\) which yields the same incentive-compatible allocation is such that \(\tau' \geq \tau\) and \(\beta' \geq \beta\). In other words, \((\tau, \beta)\) is minimal when \(y - \tau(y)\) coincides with the envelope curve of the population’s indifference curves (in the earnings-consumption space) at the allocation.

4.2 Results

The computation of the optimal tax is quite complex in this context because the population is heterogeneous in two dimensions, preferences and earning ability.\(^{10}\) It is possible, however, to obtain some results about, first, the part of the tax schedule which should be the focus of the social planner and, second, some features of the optimal tax.

The analysis is somewhat simplified if the low-skilled population is diverse enough so that, in the range of low incomes, there is no significant hole in the distribution of bundles of the low-skilled population. If there were such holes and if such holes were filled by high-skilled agents whose budget opportunities are much greater, it would be difficult to sort out, among low-income earners, those who are the worse-off in the population (for \(W_i\)). The following assumption, already introduced in Fleurbaey and Maniquet (2002a) is the simplest assumption over the primitives of the model which ensures this outcome for any size of the population.

Formally, let \(uc((y_i, c_i), w_i, R^*_i)\) denote the closed upper contour set for \(R^*_i\) at \((y_i, c_i)\):

\[
uc((y_i, c_i), w_i, R^*_i) = \{(y, c) \in [0, w_i] \times \mathbb{R}^G_+ \mid (y, c) R^*_i (y_i, c_i)\}.
\]

The assumption that is introduced now says that a high-skilled agent, when contemplating low earnings, always finds low-skilled agents who have locally similar preferences in the \((y, c)\)-space. Let \(w_m = \min_i w_i\). It is assumed throughout this section that \(w_m > 0\).

**Assumption (Low-Skill Diversity):** For every agent \(i\), and every \((y, c)\) such that \(y \leq w_m\), there is an agent \(j\) such that \(w_j = w_m\) and \(uc((y, c), w_j, R^*_j) \subseteq uc((y, c), w_i, R^*_i)\).

The inclusion of upper contour sets means that whenever agent \(i\) chooses \((y_i, c_i)\) in a budget set, there is a low-skilled agent \(j\) who is willing to choose this same bundle from the same budget set.

The first result in this section gives a simple way to compute \(\min_i W_i(z_i)\) when a minimal tax schedule is enforced.

\(^{10}\)Formally, there are more dimensions since the space of preferences is infinitely dimensional. On multi-dimensional screening, see e.g. Armstrong (1996) or Rochet and Choné (1998).
Lemma 1 Let $z$ be an incentive-compatible allocation obtainable with the minimal tax schedule $(\tau, \beta)$ and such that $z_i P_i 0$ for all $i$. Then

$$\min_i W_i(z_i) = w_m \times \min_{(y,c) \in S(\tau,\beta), y \leq w_m} \left[ 1 - \frac{\beta(c) + \tau(y)}{y} \right]$$

The intuition for this result is rather simple. Recall that for any agent $i$,

$$W_i(z_i) = \max\{w \in \mathbb{R}_+ \mid \forall (\ell,c) \in X \text{ s.t. } pc \leq w\ell, \ z_i R_i (\ell,c)\}.$$  

When $z_i P_i 0$, this is equivalent to

$$W_i(z_i) = \min\{w \in \mathbb{R}_+ \mid \exists (\ell,c) \in X \text{ s.t. } pc = w\ell, \ (\ell,c) R_i z_i\}.$$  

Translated into $(y,c)$ space, this definition is equivalent to

$$W_i(z_i) = \min \left\{ w \in \mathbb{R}_+ \mid \exists (y,c) \in \mathbb{R}^{G_i+1}_+ \text{ s.t. } pc = w \frac{y}{w_i}, \ (y,c) R_i^* (y_i, c_i) \right\}$$

$$= \min \left\{ w \frac{pc}{y} \mid (y,c) \in \mathbb{R}^{G_i+1}_+, \ (y,c) R_i^* (y_i, c_i) \right\}.$$  

For a low-skilled agent ($w_i = w_m$), one then has

$$W_i(z_i) = w_m \times \min_{(y,c) \in I_i(z_i)} \frac{pc}{y}$$

where $I_i(z_i)$ is $i$’s indifference surface in $(y,c)$ space containing $(y_i, c_i)$. From Low-Skill Diversity, one deduces that the worse-off for $W_i$ will be among the low-skilled agents, and that, for a minimal tax schedule, the subset of $S(\tau, \beta)$ such that $y \leq w_m$ coincides with the envelope curve of the low-skilled agents’ indifference surfaces at $z$. Therefore

$$\min_i W_i(z_i) = w_m \times \min_{(y,c) \in S(\tau,\beta), y \leq w_m} \frac{pc}{y}$$

One finally computes that, when $(y,c) \in S(\tau,\beta)$,

$$\frac{pc}{y} = 1 - \frac{\beta(c) + \tau(y)}{y}.$$  

From this lemma one derives a simple criterion of comparison of tax schedules, for social preferences described in Theorem 1.

Theorem 2 Consider two incentive-compatible allocations $z$ and $z'$ obtainable with two minimal tax schedules $(\tau, \beta)$ and $(\tau', \beta')$, respectively, and such that $z_i P_i 0$ and $z'_i P_i 0$ for all $i$. If social preferences satisfy Transfer Principle, Laisser-Faire, Weak Pareto, Hansson Independence and Separability, then $z$ is socially preferred to $z'$ whenever the maximal average tax rate over low incomes $y \in [0, w_m]$ is smaller in $z$:

$$\max_{(y,c) \in S(\tau,\beta), y \leq w_m} \frac{\beta(c) + \tau(y)}{y} < \max_{(y,c) \in S(\tau,\beta), y \leq w_m} \frac{\beta'(c) + \tau'(y)}{y}.$$  

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This result is straightforwardly obtained from the above lemma, by noticing that this inequality on tax rates entails that
\[ \min_i W_i(z_i) > \min_i W_i(z'_i), \]
so that one may apply Theorem 1 in order to conclude that \( z \) is socially preferred.

One limitation of this first set of results is that they deal only with minimal tax schedules. This is a serious limitation in the current framework, because one cannot imagine to have ordinary commodity taxes espousing individual indifference surfaces at corner bundles. As a concrete example, consider linear commodity taxes with two rates, a low rate on necessities and a high rate on luxuries. It is very unlikely that such a tax schedule could be minimal, because this would require that we observe some low-skilled agents consuming all combinations of goods, including bundles containing only luxuries and no necessities at all (or at least they should be indifferent between their current situation and such a corner consumption). It is therefore important to examine what can be said for non-minimal tax schedules. An exact measure of \( \min_i W_i(z_i) \) is then much harder to obtain, but the following lemma gives a useful bracket.

**Lemma 2** Let \( z \) be an incentive-compatible allocation obtainable with the (not necessarily minimal) tax schedule \((\tau, \beta)\), and such that \( z_i P_i 0 \) for all \( i \). Then
\[
\min_{(y,c) \in S(\tau,\beta), \ y \leq w_m} \left[ 1 - \frac{\beta(c) + \tau(y)}{y} \right] \leq \frac{\min_i W_i(z_i)}{w_m} \leq \min_{i : y_i \leq w_m} \left[ 1 - \frac{\beta(c_i) + \tau(y_i)}{y_i} \right].
\]

For a minimal tax schedule, the value of \( \min_i W_i(z_i) \) coincides with the lower bound. For a non minimal tax schedule, \( \min_i W_i(z_i) \) may be greater than the lower bound, because the envelope curve of individual indifference surfaces lies nowhere below but sometimes above the \( S(\tau,\beta) \) surface. By Low-Skill Diversity, however, this envelope curve necessarily contains all \((y_i, c_i)\) such that \( y_i \leq w_m \), and this yields the upper bound of the above bracket.

From this lemma, one obtains a comparison criterion which concludes with certainty when the lower bound for one allocation is greater than the upper bound for another.

**Theorem 3** Consider two incentive-compatible allocations \( z \) and \( z' \) obtainable with two (not necessarily minimal) tax schedules \((\tau, \beta)\) and \((\tau', \beta')\), respectively, and such that \( z_i P_i 0 \) and \( z'_i P_i 0 \) for all \( i \). If social preferences satisfy Transfer Principle, Laisser-Faire, Weak Pareto, Hansson Independence and Separability, then \( z \) is socially preferred to \( z' \) whenever:
\[
\max_{(y,c) \in S(\tau,\beta), \ y \leq w_m} \frac{\beta(c) + \tau(y)}{y} < \max_{i : y_i \leq w_m} \frac{\beta'(c'_i) + \tau'(y'_i)}{y'_i}.
\]
This criterion is less powerful than in the previous theorem and may be silent in some applications. As a less rigorous but convenient approximation, one may also compute the upper and lower bounds of the above lemma for the two contemplated allocations. If both bounds move up when one goes from one allocation to the other, it is very likely (but not sure) that the latter is socially better.

The criterion obtained in Theorem 3 has a practical drawback compared to Theorem 2, namely, it requires a statistical knowledge of the distribution of \((y_i, c_i)\) in at least one allocation, whereas in Theorem 2 a pure “accounting” knowledge of the tax schedules is sufficient to apply the comparison criterion. This is the cost of considering non-minimal tax schedules.

When \(w_m\) is small, the criteria highlighted above focus on the smallest net-of-taxes consumptions, since

\[
\lim_{w_m \to 0} \left[ w_m \times \min_{(y,c) \in S(\tau, \beta), y \leq w_m} \frac{pc}{y} \right] = -\tau(0) - \max_{pc \leq -\tau(0)} \beta(c),
\]

and \(-\tau(0)\) is the minimum income given to agents with zero earnings.

The above results may be more useful for practical policy decisions than characterizations of the optimal tax schedule which are usually the focus of the literature. Indeed, political constraints make it very hard to implement radical changes in tax schedules, so that reforms are typically very limited in scope and scale. For such policy decisions, a simple criterion for the comparison of arbitrary suboptimal tax schedules is more useful than a description of an unattainable ideal. It is nonetheless interesting to examine if some features of the optimum can be deduced. The following theorem provides some information about an interesting subset of the optimal taxes.

**Theorem 4** Assume that there exists an allocation \(z\) such that \(z_i P_i 0\) for all \(i\). Let \(z^*\) be an optimal incentive-compatible allocation for social preferences satisfying Transfer Principle, Laisser-Faire, Weak Pareto, Hansson Independence and Separability. It can be obtained with a tax schedule \((\tau^*, \beta^*)\) which, among all feasible tax functions, maximizes the net income of the hardworking poor, \(w_m - \tau(w_m)\), under the constraints that

\[
\begin{align*}
\frac{\beta(c)}{y} & = 0 \text{ for } c \text{ s.t. } pc = w_m - \tau(w_m), \\
\frac{\tau(y) + \beta(c)}{y} & \leq \frac{\tau(w_m)}{w_m} \text{ for all } y \in [0, w_m], (y,c) \in S(\tau, \beta), \\
\tau(y) + \beta(c) & \geq \tau(w_m) \text{ for all } (y,c) \in S(\tau, \beta), \\
\tau(0) + \beta(0) & \leq 0.
\end{align*}
\]
This theorem does not imply that all optimal tax schedules must satisfy these constraints. It only states that there is no welfare loss, when looking for the optimal allocation, in restricting attention to taxes satisfying those constraints. One sees how the social preferences defined in this paper favor the hardworking poor, who should get, in the optimal allocation, the greatest absolute amount of subsidy among the whole population (third constraint) and should be exempted from commodity taxation (first constraint). However, the taxes computed for those with a lower income than \( w_m \) also matter, as those agents must obtain at least as great a rate of subsidy as the hardworking poor (second constraint). The fourth constraint, often called the participation or individual rationality constraint in the literature, means that 0 is an attainable vector, guaranteeing that no agent is worse-off than at 0.

This result is obtained with the following kind of argument. Take any optimal incentive-compatible allocation \( z^* \) such that \( z_i^* \geq R_i \) 0 for all \( i \), and consider a corresponding tax schedule \((\tau, \beta)\). Compute \( \min_i W_i(z_i) \) and let \( W_m \) denote this quantity. Then construct another tax schedule \((\tau^*, \beta^*)\) such that
\[
\beta^*(c) = 0 \text{ for } c \text{ s.t. } pc = W_m, \\
\tau^*(y) + \beta^*(c) = \max \{w_m - W_m, \tau(y) + \beta(c)\} \text{ for all } (y, c) \in S(\tau^*, \beta^*).
\]

This new tax schedule cuts all subsidies greater than \( w_m - W_m \) but leaves taxes otherwise unchanged, so that it is still feasible. One checks that \((\tau^*, \beta^*)\) does satisfy the conditions listed in the theorem, and that for a new allocation \( \tilde{z} \) resulting from enforcing \((\tau^*, \beta^*)\), one computes \( \min_i W_i(\tilde{z}_i) = \min_i W_i(z_i^*) \). Moreover, \( \tilde{z} \) can be chosen so that it is equal to \( z^* \) except for agents whose subsidy decreases. If \( \tilde{z} \) is different from \( z \), then, necessarily it yields a resource surplus, making it possible to increase \( \min_i W_i(z_i) \) and contradicting the assumption that \( z^* \) is optimal. Therefore \( \tilde{z} \) must be equal to \( z^* \), and \((\tau^*, \beta^*)\) is an optimal tax schedule.

The above theorem interestingly suggests that non-uniform commodity taxes are not useful when applied to the population which is at the focus of social preferences. It does not imply that commodity taxes must be null (or uniform) across the board, since they may be useful in order to generate more tax revenues for the global budget.

When one restricts attention to linear commodity taxes, it may sometimes happen, for the special case of preferences which are sufficiently diverse, that non-uniform commodity taxes are useless even when the income tax is not optimal.

**Theorem 5** Restrict attention to linear commodity taxes \( \beta(c) = \sum_{k=1}^G \zeta_k p_k c_k \). Let \( \zeta^m = \)
If the tax schedule \((\tau, \beta)\) yielding allocation \(z\) is minimal, then the tax schedule 
\((\tau, \beta^m)\), with uniform tax \(\beta^m(c) = \zeta^m pc\), yields an allocation \(z'\) such that \(\min_i W_i(z'_i) = \min_i W_i(z_i)\).

This result derives from a similar argument as the previous one. The strong assumption 
here is that \((\tau, \beta)\) is minimal, which, for linear commodity taxes, means that consumer 
preferences are very diverse and spread consumers at all corners of the budget set. This 
entails that there are consumers, among those with minimal \(W_i(z_i)\), who are indifferent 
between their bundle \(z_i\) and bundles with an effective average commodity tax rate of \(\zeta^m\). 
These agents are the worst-off indeed, so that imposing \(\zeta^m\) on all goods would not worsen 
their situation, and would not put any other agent below \(\min_i W_i(z_i)\).

This result does not say that the new allocation \(z'\) is feasible, but this is very plausible, 
since it is obtained by increasing commodity taxes. It does not say either that \(z'\) is socially 
indifferent to \(z\), but if the increase of commodity taxes generates a budget surplus, then 
it is possible to change the income tax function so as to improve the allocation, retaining 
a uniform commodity tax.

In other words, although this result does not imply that the best linear commodity 
tax must be uniform in all circumstances, it does give a warning. Non-uniform commodity 
taxes may be useful only when the worse-off population does not contain individuals who 
are strongly submitted to the maximal tax rate.

5 Conclusion

The main contribution of the paper is contained in simple criteria for the comparison of 
tax schedules, as described in Theorems 2 and 3. For policy-makers, the possibility to 
relate simple comparisons of tax schedules to clear value judgments is probably at least 
as useful as a theoretical formula of the optimal tax. The fact that the computation 
of \(W_i(z_i)\) incorporates the price vector \(p\), and that this leads to criteria formulated in 
terms of the ratio \(pc/y\) (or equivalently tax/earnings), can be traced back directly to the 
Laisser-Faire axiom. Now, one can distinguish two parts in this axiom. One part says 
that consumer choices over the \(G\) goods need not be interfered with when consumers face 
producer prices \(p\), and the second part says that labor choices need not be interfered 
with when agents face the same earnings opportunities. One may criticize the latter as 
ignoring the fact that consumption-leisure preferences may be influenced by the quality of 
accessible jobs or social pressure of various sorts, so that agents who are strongly averse
to labor should receive more help than permitted under the Laisser-Faire axiom. For elaborations of alternative criteria along these lines, see Fleurbaey and Maniquet (2002b) and Fleurbaey (2003). But the former part of Laisser-Faire, which deals with consumer sovereignty over consumption goods, is much less controversial. It is this part which triggers the results about a uniform commodity tax.

The results obtained here which are favorable to a uniform commodity tax, and may be viewed as extending the Atkinson-Stiglitz (1976) conclusions, do not expunge some counter-arguments which can be found in the literature, such as those based on the insurance role of commodity taxes for goods consumed under uncertainty (Cremer and Gahvari 1995), or the impact of commodity taxes on factor prices if different goods contain different combinations of skilled and unskilled labor (Naito 1999). This model is also unable to make a difference between uniform and null commodity tax, but such difference may be important in practice if collection costs are different for commodity taxes and income tax (Boadway, Marchand and Pestieau 1994).

One limitation of this paper, which calls for further research, is that, as much of the literature on this topic, it assumes that producer prices \( p \) are fixed, due to constant returns to scale. If producer prices may be influenced by taxes, the analysis must be revised thoroughly. It has been shown in Fleurbaey and Maniquet (1996, 2001) that there is then a tension between Laisser-Faire and Separability, because the presence or absence of an agent may alter the equilibrium prices and change the evaluation of the situation of the rest of the population.

**Appendix: Proofs**

Some of the proofs below are very similar in structure to proofs in Fleurbaey and Maniquet (2002a). They are presented here when the modifications are not trivial. Let \( \text{co} \) denote the convex hull and \( \text{int} \) denote the interior.

**Lemma 3** If social preferences satisfy Transfer Principle, Weak Pareto and Hansson Independence, then for any pair of allocations \( z, z' \) and any pair of agents \( i, j \) with identical preferences \( R_0 \), such that

\[
z'_i P_0 z_i P_0 z_j P_0 z'_j R_0 0
\]

and \( z_k P_k z'_k \) for all \( k \neq i, j \), one has \( z P z' \).

**Proof:** Let \( z, z' \) satisfy the above conditions. By Hansson Independence, we can arbitrarily modify the preferences \( R_0 \) at bundles which are not indifferent to one of the four
bundles $z_i, z'_i, z_j, z'_j$. Let

$$S_i(z_i) = \{ z \in [0, 1] \times \mathbb{R}^G \mid z I_i z_i \},$$

$$S^0_i(z_i) = \{ z \in \{0\} \times \mathbb{R}^G \mid z I_i z_i \},$$

and $S_j, S^0_j$ are defined similarly. Let

$$S^*_i = \operatorname{co} \left[ S^0_i(z_i) \cup S_i(z'_i) \right],$$

$$S^*_j = \operatorname{co} \left[ S^0_j(z'_j) \cup S_j(z_j) \right].$$

These sets can be arbitrarily close (w.r.t. the Hausdorff distance) to two upper contour sets for $R_0$. We will indeed assume that there is an indifference surface for $R_0$, between $S_i(z_i)$ and $S_i(z'_i)$, such that the corresponding upper contour set is arbitrarily close to $S^*_i$, and another indifference surface, between $S_j(z_j)$ and $S_j(z'_j)$, such that the corresponding upper contour set is arbitrarily close to $S^*_j$.

One can then choose $z^1, z^2, z^3, z^4$ by

$$z_k P_k z'_k = z^3_k P_k z^2_k = z^1_k P_k z'_k$$

for all $k \neq i, j$, and

$$z^1_i P_i z_i, z^2_i \notin S^*_i, z^3_i \in \text{int} S^*_i, z_i P_i z^4_i,$$

$$z^1_j P_j z'_j, z^2_j \notin S^*_j, z^3_j \in \text{int} S^*_j, z_j P_j z^4_j,$$

$$z^1_i - z^3_i = z^2_i - z^1_j > 0, z^3_i - z^4_i = z^4_j - z^3_j > 0,$$

$$z^2_i \gg z^2_j, z^4_i \gg z^4_j.$$

By Transfer Principle, one has

$$z^2 \sim z^1 \text{ and } z^4 \sim z^3.$$

By Weak Pareto,

$$z \sim z^4 \text{ and } z^1 \sim z',$$

and by Weak Pareto and the assumption about indifference surfaces close to $S^*_i$ and $S^*_j$,

$$z^3 \sim z^2.$$

By transitivity, one concludes that $z \sim z'$. ■

**Lemma 4** If social preferences satisfy Transfer Principle, Laisser Faire, Weak Pareto, Hansson Independence and Separability, then for any pair of allocations $z, z'$ and any pair of agents $i, j$, such that

$$W_i(z'_i) > W_i(z_i) > W_j(z_j) > W_j(z'_j),$$

$z_i P_i 0$ and $z_k = z'_k$ for all $k \neq i, j$, one has $z \sim z'$. 

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Proof: See the proof of Lemma 2 in Fleurbaey and Maniquet (2002a).

Proof of Theorem 1: See the proof of Theorem 1 in Fleurbaey and Maniquet (2002a).

Proof of Lemma 1: Consider allocation $z$, and the related minimal tax schedule $(\tau, \beta)$. Since it is minimal, $S(\tau, \beta)$ is the envelope surface of the population’s indifference surfaces in $(y, c)$-space, at $z$.

We first prove the following fact: The subset

$$\{(y, c) \in S(\tau, \beta) \mid 0 \leq y \leq w_m\}$$

is the envelope surface of the indifference surfaces of agents from the $w_m$ subpopulation. Consider the set delimited by the envelope surface of all agents over this range:

$$([0, w_m] \times \mathbb{R}_+) \cap \left( \bigcup_i uc((w_i \ell_i, c_i), w_i, R_i^*) \right) = \bigcup_i \left( uc((w_i \ell_i, c_i), w_i, R_i^*) \cap ([0, w_m] \times \mathbb{R}_+) \right).$$

If the stated fact did not hold, then one would find some $(y_0, c_0)$ such that

$$(y_0, c_0) \in \bigcup_i \left( uc((w_i \ell_i, c_i), w_i, R_i^*) \cap ([0, w_m] \times \mathbb{R}_+) \right),$$

$$(y_0, c_0) \notin \bigcup_{i : w_i = w_m} \left( uc((w_i \ell_i, c_i), w_i, R_i^*) \cap ([0, w_m] \times \mathbb{R}_+) \right).$$

The first statement means that there is some $i$ such that

$$(y_0, c_0) \in uc((w_i \ell_i, c_i), w_i, R_i^*) \cap ([0, w_m] \times \mathbb{R}_+),$$

implying

$$uc((y_0, c_0), w_i, R_i^*) \subseteq uc((w_i \ell_i, c_i), w_i, R_i^*).$$

By the Low-Skill Diversity assumption, there is $j$ with $w_j = w_m$ such that

$$uc((y_0, c_0), w_j, R_j^*) \subseteq uc((y_0, c_0), w_i, R_i^*),$$

and therefore

$$uc((y_0, c_0), w_j, R_j^*) \subseteq uc((w_i \ell_i, c_i), w_i, R_i^*).$$

A consequence of this inclusion is that for any $(y, c)$ such that $(y, c)P_j^*(y_0, c_0)$, one has $(y, c)P_i^*(w_i \ell_i, c_i)$. Since

$$(y_0, c_0) \notin \bigcup_{i : w_i = w_m} \left( uc((w_i \ell_i, c_i), w_i, R_i^*) \cap ([0, w_m] \times \mathbb{R}_+) \right),$$

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one must have \((w_j \ell_j, c_j) P_j^*(y_0, c_0)\), and therefore \((w_j \ell_j, c_j) P_j^*(w_i \ell_i, c_i)\). Now, this violates the incentive-compatibility condition. We obtain a contradiction, which proves the stated fact.

Let

\[
W_m = \min_{(y, c) \in S(\tau, \beta), 0 < y \leq w_m} \left[ 1 - \frac{\beta(c) + \tau(y)}{y} \right].
\]

By the above fact, and the definition of \(S(\tau, \beta)\),

\[
W_m = w_m \min_{i: w_i = w_m} \left\{ \frac{pc}{y} \mid (y, c) \in \bigcup_{i: w_i = w_m} uc((w_i \ell_i, c_i), w_i, R_i^*) \right\},
\]

which equivalently reads

\[
W_m = w_m \min_{i: w_i = w_m} \min_{(y, c) \in uc((w_i \ell_i, c_i), w_i, R_i^*)} \left\{ \frac{pc}{y} \right\}.
\]

Now, one has, by definition:

\[
W_i(z_i) = w_i \min_{(y, c) \in uc((w_i \ell_i, c_i), w_i, R_i^*)} \left\{ \frac{pc}{y} \right\}.
\]

Therefore, \(W_m\) is the minimum value of \(W_i(z_i)\) over the \(w_m\) subpopulation. Similarly, for agents with a higher \(w\), the minimum value of \(W_i(z_i)\) is greater or equal to

\[
W_i^*(w) = w \min_{(y, c) \in S(\tau, \beta), 0 < y \leq w} \left[ 1 - \frac{\beta(c) + \tau(y)}{y} \right].
\]

It may be strictly greater than \(W_i(w)\) because, contrary to the case of \(w = w_m\) where the Low-Skill Diversity assumption applied, the envelope surface of indifference surfaces for agents with wage rate \(w > w_m\) may be above the envelope surface of all agents’ indifference surfaces over the range \([0, w]\).

Notice that, for any \(w\), either

\[
W_i(w) = w - \tau(w) - \max_{pc + \beta(c) = w - \tau(w)} \beta(c) = \min_{pc + \beta(c) = w - \tau(w)} pc
\]

or

\[
W_i^*(w) = w \min_{y_0 \in G} \frac{y_0}{pc + \beta(c) = y_0} pc \quad \text{for } y_0 < w.
\]

The latter expression is, trivially, increasing in \(w\). The former is non-decreasing since for any \(w < w'\), one has \(A(w) \subseteq A(w')\), with \(A(w) = \{ c \in \mathbb{R}_+^G \mid pc + \beta(c) \leq w - \tau(w) \}\) (recall subsection 4.1), implying

\[
\inf_{c \notin A(w)} pc \leq \inf_{c \notin A(w')} pc
\]
and therefore
\[ \min_{pc+\beta(c)=w-\tau(w)} pc \leq \min_{pc+\beta(c)=w'-(w')} pc. \]

As a consequence, \( W(w) \) is nondecreasing in \( w \), so that \( W(w) \geq W_m \), and a fortiori \( W_m \) is indeed the minimum value of \( W_i(z_i) \) over the whole population. ■

**Proof of Lemma 2:** By definition,
\[ W_i(z_i) = w_i \min \left\{ \frac{pc}{y} \mid (y, c) \in uc((w_i, \ell_i, c_i), w_i, R_i^*) \right\}, \]
and from the proof of Lemma 1,
\[ \min_i W_i(z_i) = \min_{y : y_i \leq w_m} W_i(z_i). \]

The fact that
\[ \min_{(y, c) \in S(\tau, \beta), 0 < y \leq w_m} \left[ 1 - \frac{\beta(c) + \tau(y)}{y} \right] \leq \frac{\min_i W_i(z_i)}{w_m} \]
is then an immediate consequence of the fact that the envelope surface of the agents’ indifference surfaces is nowhere below \( S(\tau, \beta) \).

The fact that
\[ \min_i W_i(z_i) \leq \min_{y : y_i \leq w_m} \left[ 1 - \frac{\beta(c_i) + \tau(y_i)}{y_i} \right] \]
is a consequence of the fact that
\[ 1 - \frac{\beta(c_i) + \tau(y_i)}{y_i} = \frac{pc_i}{y_i} \]
and that, trivially, \((y_i, c_i) \in uc((y_i, c_i), w_i, R_i^*)\) for all \( i \). ■

We need four lemmas for the proof of Theorem 4. The first three deal with the possibility of finding incentive-compatible allocations in a neighborhood of allocations satisfying some properties. We also need the following definition. Let \( f : \mathbb{R}_+ \rightarrow \mathbb{R}_G^+ \) be a correspondence such that: 1) for all \( y \geq 0 \), \( f(y) \) is a closed comprehensive\(^{11} \) subset of \( \mathbb{R}_G^+ \); 2) for all \( y < y' \), \( f(y) \subseteq f(y') \). Such a correspondence we call a “regular correspondence”.

**Lemma 5** Let \( f : \mathbb{R}_+ \rightarrow \mathbb{R}_G^+ \) be an arbitrary regular correspondence, and \( z \) an incentive-compatible (not necessarily feasible) allocation such that \( z \perp 0 \) for all \( i \).

(i) Assume that \( c_i \in \text{int} f(y_i) \) for all \( i \). Then, for any real number \( \varepsilon > 0 \) there exists an incentive-compatible allocation \( z' \) such that \((\ell_i, (1+\varepsilon)c_i) P_i z' P_i z_i \) and \( c_i' \in f(y_i') \) for all
\(^{11} \) A subset \( A \subset \mathbb{R}_G^+ \) is comprehensive when \( x \in A \) and \( x' \leq x \) implies \( x' \in A \).
By envy-freeness one has
\[ e_i(z_i, y_i) = 0 \]
for all \( i \) and \( y \). This implies \( z_i = z_i^* \).

\[ v_i(y_i, \lambda_i) = \begin{cases} 
\min \{ x \geq 0 \mid (0, x) R_i^\ast (y_i, \lambda_i) \} & \text{if } y_i \leq w_i \\
-\max \{ y \geq 0 \mid (y, 0) R_i^\ast (y, \lambda_i) \} & \text{if } y_i \leq w_i \\
-w_i - y_i / (1 + \lambda) & \text{if } y_i > w_i.
\end{cases} \]

This “value function” is continuous and strictly increasing in \( \lambda \geq 0 \) for all \( (y_i, \lambda_i) \), and represents \( i \)’s preferences on the subset of \( (y_i, \lambda_i) \) such that \( y_i \leq w_i \).

We now focus on “allocations” \( (y_i', c_i', \lambda_i')_{i=1,...,n} \) such that for some permutation \( \pi \) on \( \{1, ..., n\} \) and for some vector \( (d_1, ..., d_n) \geq 0 \), one has \( (y_i', c_i', \lambda_i') = (y_{\pi(i)}, c_{\pi(i)}, 1 + d_{\pi(i)}) \) for all \( i \). The initial allocation \( (y_i, c_i, 1)_{i=1,...,n} \) is obtained by \( \pi \) being the identity mapping and \( (d_1, ..., d_n) = 0 \). It is “envy-free” in the sense that for all \( i, k, v_i(y_i, c_i, 1) \geq v_i(y_k, c_k, 1) \). This is an immediate consequence of the fact that for any \( i, k, (y_i, c_i) R_i^\ast (y_k, c_k) \) if \( y_k \leq w_i \), and \( v_i(y_i, c_i, 1) \geq 0 > v_i(y_k, c_k, 1) \) if \( y_k > w_i \).

We can then apply the “Perturbation Lemma” in Alkan, Demange and Gale (1991, p. 1029) to conclude that there is another envy-free allocation \( (y_i', c_i', \lambda_i')_{i=1,...,n} \), for some \( \pi \) and some \( d \) such that \( 0 < d_i < \eta \) for all \( i \).

The allocation \( z' \) defined by \( z_i' = (y_i' / w_i, \lambda_i' c_i') \) for all \( i \) satisfies the desired properties. By envy-freeness one has \( v_i(y_i', c_i', \lambda_i') = v_i(y_k', c_k', \lambda_k') \) for all \( i, k \), and in particular for \( k \) such that \( \pi(k) = i \), \( v_i(y_i', c_i', \lambda_i') \geq v_i(y_{\pi(i)}, c_{\pi(i)}, 1 + d_{\pi(i)}) \) implies \( v_i(y_i, c_i, 1) \geq 0 \). Since \( v_i(y_i, c_i, 1) \geq 0 \) for all \( i \), this implies \( v_i(y_i', c_i', \lambda_i') > 0 \) for all \( i \). Therefore \( y_i' \leq w_i \) for all \( i \) and \( (y_i', c_i', \lambda_i') R_i^\ast (y_k', c_k', \lambda_k') \) for all \( i, k \) such that \( y_k \leq w_i \), which means that \( z' \) is incentive-compatible.

By construction, \( (y_i', c_i') < (y_{\pi(i)}, (1 + \eta)c_{\pi(i)}) \). Since \( (\ell_i, (1 + \varepsilon)c_i) P_i (y_j / w_i, (1 + \eta)c_j) \) for all \( i, j \) such that \( y_j \leq w_i \), it follows that \( (\ell_i, (1 + \varepsilon)c_i) P_i z_i' \) for all \( i \). In addition \( v_i(y_i', c_i', \lambda_i') > v_i(y_i, c_i, 1) \) implies \( z_i' P_i z_i \).
Finally, 

$$\sum_i (pc_i - y_i') \leq \sum_i (pc_i - y_i) + \eta \sum_i pc_i < \sum_i (pc_i - y_i) + \varepsilon.$$  

(ii) Let \( m = \max_i (pc_i - y_i) \) and \( \gamma \leq \min \{ m, \varepsilon \min_i pc_i \} \). Let \( M = \{ i \mid (pc_i - y_i) > m - \gamma \} \). For all \( i \in M \), let \( \widehat{c}_i = \left( \frac{m + \gamma}{pc_i} \right) c_i \), and let \((y_i', c_i')\) be a best bundle in the subset \( \{(y_k, \widehat{c}_k)_{k \in M}, (y_k, c_k)_{k \not\in M}\} \). For \( i \not\in M \), let \( z_i = z_i \).

The allocation \( z' \) is incentive-compatible. Indeed, for every \( i \in M \), \((y_i', c_i')\) is her best bundle in \( \{(y_k, \widehat{c}_k)_{k \in M}, (y_k, c_k)_{k \not\in M}\} \) and therefore also in \( \{(y_k, c_k)_{k \in M}, (y_k, c_k)_{k \not\in M}\} \). And since \( \widehat{c}_k < c_k \) for \( k \in M \), the fact that for any \( i \not\in M \), \((y_i', c_i')\) is a best bundle in the subset \( \{(y_k, c_k)_{k \in M}, (y_k, c_k)_{k \not\in M}\} \) entails that it is a fortiori a best bundle in \( \{(y_k, c_k)_{k \in M}, (y_k, c_k)_{k \not\in M}\} \).

For every \( i \in M \), \( z_i \geq \left( 1 - \frac{\varepsilon}{pc_i} \right) c_i > (1 - \varepsilon)c_i \). For every \( i \not\in M \), \( z_i = z_i \left( \ell_i, (1 - \varepsilon)c_i \right) \).

The fact that \( \widehat{c}_k \leq c_k \) for all \( k = 1, \ldots, n \) guarantees that \( c_k' \in f(y_k') \) for all \( k \).

Finally, 

$$\sum_i (pc_i - y_i') = \sum_{i \in M} (pc_i - y_i) + \sum_{i \not\in M} (pc_i - y_i) = \sum_{i \in M} (m - \gamma) + \sum_{i \not\in M} (pc_i - y_i) < \sum_i (pc_i - y_i).$$

\[\square\]

**Lemma 6** Let \( A \) be the set of allocations \( z \) which are feasible, incentive-compatible, and such that \( z_i P_i 0 \) for all \( i \). Let \( B \) be the set of allocations \( z \) which are feasible, incentive-compatible, and such that \( z_i R_i 0 \) for all \( i \). The set \( B \) is compact. Let \( U_i \) be a continuous representation of \( R_i \), and let \( U(z) = (U_1(z_1), \ldots, U_n(z_n)) \). The set \( \{U(z) \mid z \in B\} \) is the closure of \( \{U(z) \mid z \in A\} \) if \( A \) is non-empty.

**P roof.** See the proof of Lemma 4 in Fleurbaey and Maniquet (2002a).

**Lemma 7** If there is a feasible allocation \( z \) such that \( z_i P_i (0, 0) \) for all \( i \), then there is a feasible and incentive-compatible allocation \( z \) such that \( z_i P_i (0, 0) \) for all \( i \).

**P roof.** See the proof of Lemma 5 in Fleurbaey and Maniquet (2002a).
Lemma 8 If social preferences satisfy Transfer Principle, Laisser-Faire, Weak Pareto, Hansson Independence and Separability, then for any allocations \( z, z' \) such that \( z_i P_i 0 \) for all \( i \) and \( 0 P_i z'_i \) for some \( i_0 \), one has \( z P z' \).

Proof. Consider allocations \( z \) and \( z' \) such that \( z_i P_i 0 \) for all \( i \) and \( 0 P_i z'_i \) for some \( i_0 \). By Hansson Independence, social preferences over \( \{ z, z' \} \) are not altered if the indifference surface for \( i_0 \) at 0 is assumed to be such that \( W_{i_0}(0) < \min_i W_i(z_i) \). Let \( z'' \) be such that \( z''_{i_0} = 0 \) and for all \( i \neq i_0 \), \( z''_i P_i 0 \) and \( z''_i P_i z' \). One has \( \min_i W_i(z''_i) \leq W_{i_0}(0) < \min_i W_i(z_i) \) and by Theorem 1, \( z P z'' \). By Weak Pareto, \( z'' P z' \). Therefore \( z P z' \). ■

Proof of Theorem 4. Consider an optimal allocation \( z^* \) and a minimal tax schedule \((\tau, \beta)\) defined by the fact that \( S(\tau, \beta) \) is the envelope surface of the population’s indifference surfaces in \((y, c)\)-space, at \( z^* \). Suppose there is \( i \) such that \( 0 P_i z^*_i \). Since by Lemma 7, there is a feasible and incentive-compatible allocation \( z \) such that \( z_i P_i 0 \) for all \( i \), then by Lemma 8 \( z P z^* \) and this contradicts the assumption that \( z^* \) is optimal. Therefore one must have \( z^*_i R_i 0 \) for all \( i \). This implies in particular that \( \tau(0) + \beta(0) \leq 0 \).

The function \( W_i(.) \) is a continuous representation of individual preferences. By assumption, \( A \) is non-empty, and by Lemma 6 \( \{ \min_i W_i(z_i) \mid z \in B \} \) is the closure of \( \{ \min_i W_i(z_i) \mid z \in A \} \), implying that

\[
\sup_{z \in A} \min_i W_i(z_i) = \max_{z \in B} \min_i W_i(z_i).
\]

Assume that \( \min_i W_i(z^*_i) < \sup_{z \in A} \min_i W_i(z_i) \). Then there exists an allocation \( z \in A \) such that \( \min_i W_i(z^*_i) < \min_i W_i(z_i) \). But by Theorem 1 this implies \( z P z^* \) and this contradicts the assumption that \( z^* \) is optimal. Therefore one must have \( \min_i W_i(z^*_i) = \max_{z \in B} \min_i W_i(z_i) \). This implies that \( z^* \) is obtained by a tax schedule which, among all feasible tax schedules such that \( \tau(0) + \beta(0) \leq 0 \), maximizes \( \min_i W_i(z_i) \) at the resulting allocation \( z \). It remains to show that there is no restriction in adding the other conditions stated in the theorem, and that under these conditions maximizing \( \min_i W_i(z_i) \) is equivalent to maximizing \( w_m - \tau(w_m) \).

By the proof of Lemma 1,

\[
\min_i W_i(z^*_i) = W_m = w_m \min_{(y, c) \in S(\tau, \beta), y \leq w_m} \left[ 1 - \frac{\beta(c) + \tau(y)}{y} \right].
\]

At a laissez-faire allocation \( z^{LF} \), one has \( W_i(z^{LF}_i) \geq w_i \) for all \( i \), so that

\[
\min_i W_i(z^{LF}_i) \geq w_m.
\]
A fortiori, at the optimum,

\[ W_m = \min_i W_i(z_i^*) \geq w_m. \]

Because we restrict our attention to tax schedules such that for any \( y < y' \),

\[ \{ c \in \mathbb{R}^G_+ \mid pc + \beta(c) \leq y - \tau(y) \} \subseteq \{ c \in \mathbb{R}^G_+ \mid pc + \beta(c) \leq y' - \tau(y') \}, \]

one may define two mappings \( \overline{\gamma}(c) \) and \( \underline{\gamma}(c) \) such that for every \( c \),

\[ \overline{\gamma}(c) = \max \{ y \mid pc + \beta(c) \geq y - \tau(y) \}, \]
\[ \underline{\gamma}(c) = \min \{ y \mid pc + \beta(c) \leq y - \tau(y) \}. \]

When \( \underline{\gamma}(c) < \overline{\gamma}(c) \), then \( pc + \beta(c) = y - \tau(y) \) for all \( \underline{\gamma}(c) \leq y \leq \overline{\gamma}(c) \), implying \( \tau(y) - \tau(\overline{\gamma}(c)) = y - \underline{\gamma}(c) \).

Let a new tax schedule be defined by

\[ \tau^*(y) = \tau(y) - \tau(w_m) + w_m - W_m, \]
\[ \beta^*(c) = \begin{cases} \beta(c) + \tau(w_m) - w_m + W_m & \text{for } c \text{ such that } \tau(\overline{\gamma}(c)) + \beta(c) > w_m - W_m \\ w_m - W_m - \tau^*(pc + w_m - W_m) & \text{for } c \text{ such that } \tau(\overline{\gamma}(c)) + \beta(c) \leq w_m - W_m. \end{cases} \]

In particular, one has \( \tau^*(w_m) = w_m - W_m \). Take any \( c \) such that \( pc = W_m \). By construction of \( W_m \), \( \tau(\overline{\gamma}(c)) + \beta(c) \leq w_m - W_m \), so that

\[ \beta^*(c) = w_m - W_m - \tau^*(pc + w_m - W_m) = w_m - W_m - \tau^*(w_m) = 0. \]

One also checks that, by construction, for all \((y, c) \in S(\tau^*, \beta^*)\),

\[ \tau^*(y) + \beta^*(c) = \max \{ w_m - W_m, \tau(y) + \beta(c) \}. \]

This tax schedule is feasible, because it cuts all subsidies greater than a constant, \( W_m - w_m \geq 0 \), so that no agent’s tax may decrease (and no subsidy increase), even after they adjust their choice. Moreover, for all \((y, c) \in S(\tau^*, \beta^*)\) with \( 0 < y \leq w_m \), either \((y, c) \in S(\tau, \beta)\) and

\[ \frac{pc}{y} = \frac{y - \tau(y) - \beta(c)}{y} \geq \frac{W_m}{w_m}, \]

or \((y, c) \notin S(\tau, \beta)\) and

\[ \frac{pc}{y} = \frac{y - (w_m - W_m)}{y} \geq \frac{W_m}{w_m}. \]
This implies that
\[
\min_{(y,c) \in S(\tau^*, \beta^*)}, 0 < y \leq w_m \frac{pc}{y} \geq \frac{W_m}{w_m}.
\]

On the other hand, since \((\tau^*, \beta^*)\) reduces subsidies, one has, for all \(y, c\),
\[
\{ c \in \mathbb{R}^+_+ | pc + \beta^*(c) \leq y - \tau^*(y) \} \subseteq \{ c \in \mathbb{R}^+_+ | pc + \beta(c) \leq y - \tau(y) \},
\]
implying
\[
\min_{(y,c) \in S(\tau^*, \beta^*)}, 0 < y \leq w_m \frac{pc}{y} \leq \min_{(y,c) \in S(\tau^*, \beta^*)}, 0 < y \leq w_m \frac{pc}{y} - \frac{W_m}{w_m}.
\]
Therefore,
\[
\min_{(y,c) \in S(\tau^*, \beta^*)}, 0 < y \leq w_m \frac{pc}{y} = \min_{(y,c) \in S(\tau^*, \beta^*)}, 0 < y \leq w_m \frac{pc}{y} - \frac{W_m}{w_m}.
\]

The tax schedule \((\tau^*, \beta^*)\) need not be minimal. Let \(z^{**}\) be an allocation obtained with \((\tau^*, \beta^*)\), and chosen so that \(z_i^{**} = z_i^*\) for all \(i\) such that \(\tau(y_i^*) + \beta(c_i^*) \geq w_m - W_m\). One easily checks that
\[
\min_i W_i(z_i^{**}) = W_m.
\]

The allocation \(z^{**}\) has been constructed so that for every \(i\), either \(z_i^{**} = z_i^*\) and \(\tau^*(y_i^{**}) + \beta^*(c_i^{**}) = \tau(y_i^*) + \beta(c_i^*)\), or \(\tau^*(y_i^{**}) + \beta^*(c_i^{**}) > \tau(y_i^*) + \beta(c_i^*)\). Suppose there is \(i\) such that \(\tau^*(y_i^{**}) + \beta^*(c_i^{**}) > \tau(y_i^*) + \beta(c_i^*)\). Then one has \(\sum_i [\tau^*(y_i^{**}) + \beta^*(c_i^{**})] > 0\), meaning that \(\sum_i pc_i^{**} < \sum_i y_i^{**}\). By Lemma 5, this inequality contradicts the fact that \(z^{**}\) maximizes \(\min_i W_i\) over the set of feasible and incentive-compatible allocations. Therefore there is no \(i\) such that \(\tau^*(y_i^{**}) + \beta^*(c_i^{**}) > \tau(y_i^*) + \beta(c_i^*)\), and for all \(i, z_i^{**} = z_i^*\). This means that \(\tau^*\) implements \(z^*\).

By construction, for all \((y, c) \in S(\tau^*, \beta^*)\),
\[
\tau^*(y) + \beta^*(c) \geq w_m - W_m = \tau^*(w_m).
\]

Moreover, for all \(y \in (0, w_m], (y, c) \in S(\tau^*, \beta^*)\),
\[
\frac{pc}{y} \geq \frac{W_m}{w_m}, \quad \frac{y - \tau^*(y) - \beta^*(c)}{y} \geq \frac{w_m - \tau^*(w_m)}{w_m}, \quad \frac{\tau^*(y) + \beta^*(c)}{y} \leq \frac{\tau^*(w_m)}{w_m}.
\]

Since \(W_m = w_m - \tau^*(w_m)\), maximizing \(W_m\) is equivalent to maximizing \(w_m - \tau^*(w_m)\).
Proof of Theorem 5. Since the tax schedule \((\tau, \beta)\) yielding allocation \(z\) is assumed to be minimal, there is an agent \(i_0\) with \(w_{i_0} = w_m\) such that \((y_{i_0}, c_{i_0}) \ I^*_i (y'_{i_0}, c'_{i_0})\) for some \((y'_{i_0}, c'_{i_0})\) such that \(y'_{i_0} > 0\), \(\beta(c'_{i_0}) = \zeta^m pc'_{i_0}\), and

\[
\frac{pc'_{i_0}}{y'_{i_0}} = \frac{y'_{i_0} - \tau(y'_{i_0}) - \zeta^m pc'_{i_0}}{y'_{i_0}} = \frac{\min_i W_i(z_i)}{w_m}.
\]

The first equality implies

\[
\frac{pc'_{i_0}}{y'_{i_0}} = \frac{y'_{i_0} - \tau(y'_{i_0})}{(1 + \zeta^m)y'_{i_0}}
\]

and the second means that

\[
\frac{pc'_{i_0}}{y'_{i_0}} = \min_{0 < y \leq w_m} \frac{y - \tau(y)}{(1 + \zeta^m)y}.
\]

Under the tax schedule \((\tau, \beta^m)\), \((y'_{i_0}, c'_{i_0})\) is a best bundle for \(i_0\) in his budget set. Let the other agents \(i \neq i_0\) choose some \(z'_{i}\) in their budget set. For all \((y, c) \in S(\tau, \beta^m)\) such that \(y > 0\),

\[
\frac{pc}{y} = \frac{y - \tau(y) - \zeta^m pc}{y}
\]

implying

\[
\frac{pc}{y} = \frac{y - \tau(y)}{(1 + \zeta^m)y} \geq \min_{0 < y \leq w_m} \frac{y - \tau(y)}{(1 + \zeta^m)y} = \frac{\min_i W_i(z_i)}{w_m}.
\]

Since the agents’ indifference surfaces lie nowhere below \(S(\tau, \beta^m)\), this implies that

\[
\min_i W_i(z_i) \leq \min_i W_i(z'_{i}).
\]

On the other hand, \(\beta^m \geq \beta\) and therefore

\[
\min_i W_i(z_{i}) \geq \min_i W_i(z'_{i}).
\]

As a consequence,

\[
\min_i W_i(z'_{i}) = \min_i W_i(z_{i}).
\]

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